Martingales

Example/motivation:

Let $X_0, X_1, X_2, \ldots$ be a s'ce of RVs on Ω

Z - any RV on Ω

We start on day zero & gain new information every day & observe the RV X_n on day n.

The information we have on day n can be represented by a σ-alg $\mathcal{F}_n$.

We'll have $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots$

e.g. if the only info we gain on day n is the value of X_n, we'll have $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$.

E.g.: X_n - value of a stock on day n.

Say we're on day n. What is the best guess for X_{n+1}?

i.e. $E(X_{n+1} \mid \text{info we have on day } n)$

$E(X_{n+1} \mid \mathcal{F}_n)$.

assuming no insider info, dividends, etc,

best guess for $E(X_{n+1} \mid \mathcal{F}_n)$ should be X_n.

($\mathcal{F}_n$ doesn't need to be $\mathcal{F}_n = \sigma(X_0, X_1, X_2 \ldots X_n)$.

e.g. maybe we're also keeping track of other stocks daily,

say $Y_0, Y_1, Y_2 \ldots$

$Z_0, Z_1, Z_2 \ldots$

so $\mathcal{F}_n = \sigma(X_0, \ldots, X_n, Y_0, \ldots, Y_n, Z_0, \ldots, Z_n)$.)

Defn: A stochastic process is a collection of RVs on $(\Omega, \mathcal{F}, P)$

If $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}$ are sub σ-algebras, then $\{\mathcal{F}_n\}_{n\geq 1}^{\infty}$ is called a filtration.

The process $\{X_n\}_{n\geq 1}^{\infty}$ is adapted to the filtration $\{\mathcal{F}_n\}_{n\geq 1}^{\infty}$ if X_n is $\mathcal{F}_n$-m'ble $\forall\, n \geq 1$.

(think of it as, if have the info $\mathcal{F}_n$, then know the value of X_n).

Ex: $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Then $\{X_n\}_{n\geq 1}^{\infty}$ is always adapted to $\{\mathcal{F}_n\}_{n\geq 1}^{\infty}$

Def: A stochastic process $X = \{X_n\}_{n\geq 1}^{\infty}$ is a martingale wrt the filtration $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 1}^{\infty}$ if

1) X_n is adapted to $\mathcal{F}$
2) $X_n \in L^1(P)$ $\forall n \geq 1$
3) $E[X_{n+1} | \mathcal{F}_n] = X_n$ a.s. $\forall n \geq 1$.

X is a submartingale if

3') $E[X_{n+1} | \mathcal{F}_n] \geq X_n$ a.s. $\forall n \geq 1$

X is a supermartingale if

3'') $E[X_{n+1} | \mathcal{F}_n] \leq X_n$ a.s. $\forall n \geq 1$

Rmk: Martingale "the best guess for X_{n+1} given present info is X_n."
Super vs sub: If a supermartingale agrees w/ a "similar" martingale now, then in the past it would tend to be larger

Rmk: martingale $\Leftrightarrow$ superm & subm

Ex: Simple RW. $X_1, X_2, \ldots$ iid $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$
(example of fair game)

$S_k = X_1 + \cdots + X_k$ - cumulative fortune after k steps.

$$E[X_k | X_1, \ldots, X_{k-1}] \overset{\text{indep'ce}}{=} E X_k = 0$$

so $E[S_k | X_1, \ldots, X_{k-1}] = S_{k-1}$ so $S = \{S_n\}_{n\geq 1}^{\infty}$ is a martingale wrt the filtration $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 1}^{\infty}$ w/ $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$.

Rmk: Only needed indep'ce & $E X_i = 0$ $\forall i$, to get S martingale wrt $\mathcal{F}$. No need for identically distributed

Ex: If $\mathcal{F}$ is a filtration & $Y \in L^1(P)$, $X_n := E[Y | \mathcal{F}_n]$ is a martingale (these are called Doob martingales).

These are called "closed" by le Gall.

Def: A stoch. pr. $\{A_i\}_{i\geq 1}^{\infty}$ is previsible wrt filtration $\mathcal{F}=\{\mathcal{F}_n\}_{n\geq 1}^{\infty}$ if A_n is $\mathcal{F}_{n-1}$ m'ble $\forall n\geq 1$, where $\mathcal{F}_0=\{\emptyset,\Omega\}$.

Rmk: Previsible is also called predictable.

Ex: $X_1, X_2, X_3, \ldots$ indep. $EX_i=0$ (fair game).

$S_n = X_1+\dots+X_n$ – winnings after n'th round.

Of course S_n – martingale

What if we change the bets every day?

If we bet A_n on day n, our overall winnings will be

$$Y_n = Y_{n-1} + A_n X_n = Y_{n-1} + A_n(S_n - S_{n-1})$$
$$= \underset{\substack{\uparrow \\ \text{some starting amount}}}{y_0} + \sum_{i=1}^{n} A_i(S_i - S_{i-1})$$

How much we bet can only be based on info we had up to that point so $A_n \in \mathcal{F}_{n-1}$, i.e. $A=\{A_n\}_{n\geq 1}^{\infty}$ is previsible.

More generally

Martingale transforms

Let $\{S_n\}_{n=1}^{\infty}$ be a martingale wrt filtration $\mathcal{F}=\{\mathcal{F}_i\}_{i\geq 1}^{\infty}$.

Let $S_0=0$, y_0 – const & $A=\{A_n\}_{n\geq 1}^{\infty}$ previsible pr wrt $\mathcal{F}$.

Define pr Y by $Y_n := y_0 + \sum_{j\geq 1}^{n} A_j(S_j - S_{j-1})$ $\forall n\geq 0$.

Exercise: Y is a martingale

Y is called a martingale transform of S.

Le Gall writes if $(A_n)_n$ previsible, $(X_n)_n$ is a Martingale,

$$(A\bullet X)_n = \sum_{i=1}^{n} A_i(X_i - X_{i-1}) \text{ "=" } \int A\,dX$$

Prop: If (X_n) is a super(sub)martingale and (H_n) previsible and nonnegative Then $(H \bullet X)_n$ is a super (sub) martingale.

Example: $\Omega = \{-1,+1\}^{\mathbb{Z}_+}$, $P = \mu^{\otimes \mathbb{Z}_+}$ (product meas.)
let $Y_n(\omega) = \omega_n$ represent the outcome of the game.
If its fair $E[Y_n(\omega)] = 0$.

Then $S_n = \sum_{i=1}^{n} Y_i$ is a martingale.

If $H_n = f(Y_1, \cdots, Y_{n-1})$ is a previsible "bet" then if you win you gain $H_n(S_n - S_{n-1}) = H_n$ and if you lose you gain $H_n(S_n - S_{n-1}) = -H_n$. Thus your net winnings after n games is $(H \bullet S)_n$.

Another transform:

Prop: Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex, and let X_n be $\mathcal{F}_n$ adapted.

1) If (X_n) is a martingale, and $E|\varphi(X_n)| < \infty$ then $\varphi(X_n)$ is a submartingale.

2) If $(X_n)_{n \in \mathbb{Z}_+}$ is a submartingale and φ is increasing then $\varphi(X_n)$ is a submartingale.

Pf: 1) $E[\varphi(X_{n+1}) \mid \mathcal{F}_n] \geq \varphi\left(E[X_{n+1} \mid \mathcal{F}_n]\right) = \varphi(X_n)$ (★1)

This needs the L^1 condition to be defined. If $\varphi : \mathbb{R} \to \mathbb{R}_+$ as in Le Gall, then can drop $E|\varphi(X_n)| < \infty$. Recall that we defined $E[X \mid \mathcal{F}] := \lim_{n\to\infty} E[X \wedge n \mid \mathcal{F}]$ if $X \geq 0$.

2) Similarly, if X_n is only a submartingale, the last step has $E[X_{n+1} \mid \mathcal{F}_n] \geq X_n$ then we get (★1).

Properties of martingales

Lemma: If X is a subm. wrt filtration $\mathcal{F}$, then it is a subm. wrt the filtration generated by X itself, i.e.

$$\mathbb{E}[X_{n+1} \mid X_1, \ldots, X_n] \geq X_n \text{ a.s.}$$

Pf: Know $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \geq X_n$ $\forall n$.

X_n – $\mathcal{F}_n$-m'ble $\forall n$ & $\mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n \Rightarrow \sigma(X_1, \ldots, X_n) \subseteq \mathcal{F}_n$.

$$\mathbb{E}\left[\mathbb{E}[X_{n+1}|\mathcal{F}_n] \mid X_1, \dots, X_n\right] \geq \mathbb{E}[X_n | X_1, \dots, X_n]$$

$$\mathbb{E}[X_{n+1} | X_1, \dots, X_n] \geq X_n$$

$\triangle$.

Lemma: If X is a martingale & φ is convex s.t. $\varphi(X_n) \in L^1(P)$ $\forall n$ then $\varphi(X)$ is a subm. If X subm & φ - non-decr. convex, w/ $\varphi(X_n) \in L^1(P)$ $\forall n$, then $\varphi(X)$ is also subm.

Pf: Conditional Jensen

$$\mathbb{E}[\varphi(X_{n+1})|\mathcal{F}_n] \geq \varphi\, \mathbb{E}[X_{n+1}|\mathcal{F}_n] = \varphi(X_n). \text{ if } X \text{ martingale}$$

If X - subm & φ - non decr, then

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$$

so $\varphi(\mathbb{E}[X_{n+1}|\mathcal{F}_n]) \geq \varphi(X_n)$ $\triangle$

Cor: If X - martingale, then X^+, $|X|^p$, e^X are subm. if they stay in $L^1(P)$ $\forall n$.

If X - subm., then X^+, e^X also subm if in $L^1(P)$ $\forall n$.

If X - subm, $|X|$ might not be.

e.g. $X_k := -\frac{1}{k}$.

Thm (Doob's decomposition thm).

Any submartingale X can be written as

$X_n = Y_n + Z_n$ where Y is a martingale & Z is a non-neg. previsible a.s. increasing process w/ $Z_n \in L^1(P)$ $\forall n$

Pf: Set $X_0 := 0$ & $d_j = X_j - X_{j-1}$ so $X_n = d_1 + \dots + d_n$.

Suppose have Y_n, Z_n, want to find out Y_{n+1}, Z_{n+1}.

Let $e_{n+1} = Y_{n+1} - Y_n$, $f_{n+1} = Z_{n+1} - Z_n$. Have $d_{n+1} = e_{n+1} + f_{n+1}$

Must have Z_{n+1} is $\mathcal{F}_n$-m'ble & $\mathbb{E}(e_{n+1}|\mathcal{F}_n) = 0$

$$\mathbb{E}(d_{n+1}|\mathcal{F}_n) = \mathbb{E}(e_{n+1}|\mathcal{F}_n) + \mathbb{E}(f_{n+1}|\mathcal{F}_n)$$

$$\mathbb{E}[d_{n+1}|\mathcal{F}_n] = 0 + f_{n+1}$$

So should take $f_{n+1} = E(d_{n+1} | \mathcal{F}_n)$
$$e_{n+1} = d_{n+1} - E(d_{n+1} | \mathcal{F}_n).$$

Thus $$Z_n = \sum_{k=1}^{n} E(d_k | \mathcal{F}_{k-1})$$
$$Y_n = X_n - Z_n.$$

I have a different way of seeing it since its not clear a priori why we should look at $d_n = X_n - X_{n-1}$

Submartingales increase on average. So one wants to look at the "excess" $X_n - E[X_n | \mathcal{F}_{n-1}]$ that prevents X_n from being a martingale. Notice,

$$X_n = Y_n + Z_n \qquad E[X_n | \mathcal{F}_{n-1}] = Y_{n-1} + Z_n$$

$$\Rightarrow X_n - E[X_n | \mathcal{F}_{n-1}] = Y_n - Y_{n-1} \qquad (\text{the } Z_n \text{ cancel})$$

$$\Rightarrow Y_n - \cancel{Y_0}^{\,0} = \sum_{i=1}^{n} X_n - E[X_n | \mathcal{F}_{n-1}] \qquad (\text{just set } Y_0 = 0)$$

Then $$Z_n = X_n - Y_n = \cancel{X_n} - \cancel{X_n} + E[X_n | \mathcal{F}_{n-1}] - X_{n-1} + E[X_{n-1} | \mathcal{F}_{n-2}] - X_{n-2}$$
$$= E[X_n - X_{n-1} | \mathcal{F}_{n-1}] + E[X_{n-1} - X_{n-2} | \mathcal{F}_{n-2}] + \cdots$$

Its clear Z_n is previsible and indeed increasing.

Remark: We have shown $X_n \geq Y_n$ a martingale.
Can we show $X_n \leq \tilde{Y}_n$ another martingale?

Kr decomp. thm

Defn: $\{X_i\}_{i\geq 1}^{\infty}$ is bdd in $L^r(P)$ if $\sup_n \|X_n\|_r < \infty$.

Thm: (Krickeberg's decomposition).
If X is a subm. bdd in $L^1(P)$ then can write it as
$X_n = Y_n - Z_n$ w/ Y_n - mart., Z_n - non-neg. superm.

Pf: Set $Y_n = \lim_{m\to\infty} E[X_m | \mathcal{F}_n]$.

$E[X_{m+1} | \mathcal{F}_n] \geq E[X_m | \mathcal{F}_n]$ a.s.

$\Rightarrow$ the limit exists.

Let $m \geq n$. $E[X_{m+1} | \mathcal{F}_n] \geq X_n \Rightarrow Y_n > X_n \Rightarrow Z_n = Y_n - X_n \geq 0$
(use towering property)

Check Y_n - martingale 1) Adapted ✓

2) $Y_n \in L^1$: $E|Y_n| = E\left[\lim_{m\to\infty} |E[X_m | \mathcal{F}_n]|\right] \leq \lim_m E|X_m| \overset{\text{subm}}{=} \sup_m E|X_m| \overset{X_n \text{ bdd in } L^1(P)}{<} \infty$

$E[Y_{n+1} | \mathcal{F}_n] = E\left[\lim_{m\to\infty} \underbrace{E[X_{m+1} | \mathcal{F}_{n+1}]}_{\text{increasing in } m} \,\middle|\, \mathcal{F}_n\right] \underset{\text{Cond'l MCT}}{=} \lim_{m\to\infty} E[E[X_{m+1} | \mathcal{F}_{n+1}] | \mathcal{F}_n]$

$= \lim_{m\to\infty} E[X_{m+1} | \mathcal{F}_n] = Y_n$.

Y_n is a martingale $\Rightarrow E[Z_n | \mathcal{F}_{n-1}] = Y_{n-1} - E[X_n | \mathcal{F}_{n-1}] \leq Y_{n-1} - X_{n-1} = Z_{n-1}$

Rmk: Y & Z are L^1-bdd.

Pf: $Y_n = X_n + Z_n$ $\qquad Z_n \geq 0$

$E|Y_n| \leq \underbrace{\sup_k E|X_k|}_{\text{indep of } n, \text{ finite}} + E Z_n$.

$|E Z_n| = |E(Y_n - X_n)| = |E Y_1 - E X_n| \leq |E Y_1| + \underbrace{\sup_k E|X_k|}_{\text{indep of } n, \text{ finite}}$.

△

Doob + Krickeberg: submartingales are bounded above and below by martingales.

Thm: $\forall$ bdd L^1-bdd subm is bdd above & below by martingales.

Pf: Doob — $\forall$ subm bdd below by mart.
Krickeberg — $\forall$ L^1-bdd subm bdd above by mart. $\triangle$

Stopping times

Defn: A stopping time wrt a filtration $\mathcal{F}$ is a RV $T: \Omega \to \mathbb{N} \cup \{\infty\}$ s.t. $\{T=k\} \subseteq \mathcal{F}_k$ $\forall k \in \mathbb{N}$
(equiv. $\{T \le k\} \subseteq \mathcal{F}_k$ $\forall k \in \mathbb{N}$)

$\{T = +\infty\} = \bigcap_{n \in \mathbb{Z}_+} \{T \le n\}^c$ Each of these are $\mathcal{F}_n$ meas.
$\Rightarrow \{T = +\infty\} \in \mathcal{F}_\infty$

Ex 1) $T = n$ a.s is a stopping time since $\{T=n\} \Delta \Omega = \emptyset$

2) If $A \in \mathcal{B}(\mathbb{R})$, $T_A = \inf\{n : X_n \in A\}$ is a stopping time.

$\{T_A = k\} = \{X_1 \notin A, \dots, X_{k-1} \notin A, X_k \in A\} \in \mathcal{F}_k$

Actually, every stopping time can be expressed as a 1st visit time
If T - stopping time, define
$X_j(\omega) := \mathbb{1}_{\{T=j\}}(\omega)$.
Then T is the 1st time the stoch. pr. $\{X_j\}_{j \ge 1}^{\infty}$ visits $A = \{1\}$

This may not be that useful in general.

3) $L_A = \sup\{n : X_n \in A\}$ is not in general.

$\{L_A = k\} = \{X_k \in A, X_{k+1} \notin A, \dots\}$

4) S, T stopping times $\{\max(S,T) = k\} = \{S = k, T \leq k\} \cup \{T = k, S \leq k\} \in \mathcal{F}_k$

<u>Hint:</u> Check that if $\{T_i\}_{i=1}^{n}$ are stopping times then so are $T_1 + \dots + T_n$, $\min\limits_{1 \leq i \leq n} T_i$, $\max\limits_{1 \leq i \leq n} T_i$

Stopped σ-algebras

Rem: The whole point of $\mathcal{F}_T$ is to find a σ algebra st $X_T := X_{T(\omega)}(\omega)$ is meas. wrt to it. One option is to consider $\sigma(T)$, but if $T = \text{cos}$, then $\sigma(T)$ is trivial. we want $\mathcal{F}_c$ instead.

$$\mathcal{F}_T := \{ A \in \mathcal{F} \mid A \cap \{T \le k\} \subseteq \mathcal{F}_k \ \forall k \in \mathbb{N} \}.$$

(note this is not a random σ-alg, you dont choose $T=k$ & pick $\mathcal{F}_k$)

Thm: $\mathcal{F}_T$ - σ-alg

Lemma: 1) If S, T - stopping times st $P(S<\infty) = P(T<\infty) = 1$, & $S \le T$ a.s., then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

2) $E[Y \mid \mathcal{F}_T] \mathbb{1}_{T=n} = E[Y \mid \mathcal{F}_n] \mathbb{1}_{T=n} \quad \forall Y \in L^1(P), \ \forall n \ge 1.$

Pf 1) If $A \in \mathcal{F}_S$, then $A \cap \{T=n\} = \bigcup_{k=0}^{\infty} (A \cap \{S=k\}) \cap \{T=n\}$

$= \bigcup_{k=0}^{n} (A \cap \{S=k\}) \cap \{T=n\} \in \mathcal{F}_n$ (since $S \le T$ we can restrict the sum)

$\Rightarrow A \in \mathcal{F}_T$

2) Skip.

The random variables $E[Y \mid \mathcal{F}_T]$ and $E[Y \mid \mathcal{F}_n]$ are different, but they agree on the set $\mathbb{1}_{\{T=n\}}$.

☆ This is a good HW problem. Let $B \in \mathcal{F}_T$

$E\left[E[Y \mid \mathcal{F}_T] \mathbb{1}_{\{T=n\}} \mathbb{1}_B \right]$ $\qquad$ $B \cap \{T=n\} \in \mathcal{F}_n$ by defn.

$B \cap \{T=n\} \cap \{T=k\}$ is nonempty only for $k=n$ in which case its in $\mathcal{F}_n$

$=$ ☆1

So $B \cap \{T = n\} \in \mathcal{F}_T$ as well. Then by the defining property

$$\star 1 = E[Y \mathbb{1}_{B \cap \{T=n\}}] = E[E[Y | \mathcal{F}_n] \mathbb{1}_{\{T=n\}} \mathbb{1}_B]$$

<u>Prop</u> If S, T stopping times, $\mathcal{F}_{S \wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$

$\mathcal{F}_{S \wedge T} \subset \mathcal{F}_S \cap \mathcal{F}_T$ from 1)

If $A \in \mathcal{F}_S \cap \mathcal{F}_T$ $\quad A \cap \{S \wedge T \leq n\} = A \cap (\{S \leq n\} \cup \{T \leq n\})$

$$= \underbrace{(A \cap \{S \leq n\})}_{\in \mathcal{F}_n} \cup \underbrace{(A \cap \{T \leq n\})}_{\in \mathcal{F}_n}$$

$\Rightarrow A \in \mathcal{F}_{S \wedge T}$

Note: $\quad A \cap \{S \leq n\} \in \mathcal{F}_n$. $\{S \leq n\} = \{S \leq n\} \cap A \sqcup (\{S \leq n\} \cap A^c)$

$\Rightarrow \{S \leq n\} \cap A^c = \{S \leq n\} \setminus (\{S \leq n\} \cap A) \in \mathcal{F}_n$

$\Rightarrow A^c \in \mathcal{F}_S$. This can be used to show $\mathcal{F}_S$ is a σ-algebra.

☆ Q: $\quad A \in \mathcal{F}_S \Rightarrow A \cap \{S \leq n\} \in \mathcal{F}_n$. Does this also imply $A \cap \{S > n\} \in \mathcal{F}_n$

I cant say off the top of my head.

Prop Let (Y_n) be $\mathcal{F}_n$ adapted and let T be a stopping time.

Then Y_T is well defined on $\{T<\infty\}$ and is meas. wrt $\mathcal{F}_T$.

Pf: $\{Y_T \in B\} \cap \{T=n\} = \{Y_n \in B\} \cap \{T=n\}$

$\{Y_n \in B\} \in \mathcal{F}_n$, $\{T=n\} \in \mathcal{F}_n$

Thm: (Optional Stopping, easy) Let $(X_n)_{n\in\mathbb{Z}}$ be a martingale (or super)

Let T be a stopping time, then $(X_{n\wedge T})$ is a martingale (or super)

a) If $T \le K$ a.s. $E[X_T] = E[X_0]$

b) If $|X_n| \le K$ a.s., and $T<\infty$ a.s., $E[X_T] = E[X_0]$

Pf: Let $H_n = 1_{\{T \geq n\}}$ $\mathcal{F}_{n-1}$ meas.

$$X_{n\wedge T} = \sum_{i=1}^{n\wedge T} X_i - X_{i-1} + X_0 = X_0 + \sum_{i=1}^{n} 1_{\{i \le T\}} (X_i - X_{i-1})$$

$$= X_0 + (H \bullet X)_n$$

For a) $$\lim_{n\to\infty} E[X_{n\wedge T}] = E[X_{K\wedge T}] = E[X_0]$$

b) $$\lim_{n\to\infty} E[X_{n\wedge T}] \overset{\text{bdd}}{=} E[X_T] = E[X_0]$$

Optional stopping theorem

Consider a fair game. E.g. $X_1, X_2, \ldots$ indep, $EX_i = 0\ \forall i$,
$S_n = X_1 + \cdots + X_n$.
$ES_n = 0$, so if playing for n steps your expected winnings are 0.
What if you devise some strategy for when you stop instead of stopping after n steps? You have to stop at a stopping time (can't use info from the future).
Can you come up w/ a strategy T s.t. $ES_T > 0$?
No (Saw w/ random walks, $ES_N = EX_1 EN = 0\ \forall$ stopping time N)

Thm: (Doob's optional stopping thm).
Suppose S, T are a.s. bdd stopping times s.t. $S \le T$ a.s.
If X is a subm, then $E[X_T | \mathcal{F}_S] \geq X_S$ a.s.
supm $\leq$
martingale $=$

pf: Let $k \in \mathbb{N}$ be s.t. $S \le T \le k$ a.s. (k - const).
X - subm.
Write $X_n = d_1 + d_2 + \cdots + d_n$ (i.e. $d_k := X_k - X_{k-1}$, $\forall k \in \mathbb{N}$ w/ $X_0 = 0$).
X - subm. $\Rightarrow E[d_j | \mathcal{F}_{j-1}] \geq 0$ a.s. $\forall j$

$$X_T = \sum_{j=1}^{k} d_j \mathbb{1}_{j \le T} \quad \text{ditto for } X_S.$$

ETS $E[X_T - X_S; A] \geq 0 \quad \forall A \subseteq \mathcal{F}_S$.

$$E[X_T - X_S; A] = \sum_{j=1}^{k} E[d_j \mathbb{1}_{S < j \le T}; A]$$

$$= \sum_{j=1}^{k} E[d_j \mathbb{1}_{\{S < j \le T\} \cap A}]$$

$$= \sum_{j=1}^{k} \mathbb{E}\Big[\mathbb{E}\big[d_j \mathbb{1}_{\{S<j\le T\}\cap A} \,\big|\, \mathcal{F}_{j-1}\big]\Big]$$

$$\{S<j\le T\}\cap A = \underbrace{\{S<j\}\cap A}_{\in \mathcal{F}_{j-1}} \cap \underbrace{\{T<j\}^c \cap A}_{\in \mathcal{F}_{j-1}}$$

$$= \sum_{j=1}^{k} \mathbb{E}\Big[\mathbb{E}[d_j|\mathcal{F}_{j-1}]\, \mathbb{1}_{\{S<j\le T\}\cap A}\Big] \geq 0. \quad \triangle$$

Cor: If T is a stopping time wrt $\mathcal{F}$ & X submartingale (supermartingale), then so is $\{X_{T\wedge n}\}_{n\ge1}^{\infty}$ wrt $\{\mathcal{F}_{T\wedge n}\}_{n\ge1}^{\infty}$.

~~Pf:~~ apply prev thm w/ $T = T\wedge n+1$ & $S = T\wedge n$ $\triangle$

Read Section 8.4 to see how to obtain the Random walk results from earlier via martingale theory. Especially Thm 8.34.

St. Petersberg paradox

Recall $\Omega = \{-1,+1\}^{\mathbb{Z}_+}$, $P = \mu^{\otimes \mathbb{Z}_+}$ (product meas.)

Let $Y_n(\omega) = \omega_n$ represent the outcome of the game.

If its fair $E[Y_n(\omega)] = 0$.

Let H_0 be initial bet, $H_n = 2 H_{n-1}$ (double)

Let $T = \inf\{n : Y_n = 1\}$ (quits)

Net winnings after n games is $(H \bullet S)_n$.

We care about $(H \bullet S)_T = W_T$

$$W_T = \sum_{i=0}^{T-1} H_i Y_i + H_T Y_T \quad \text{on } T < \infty$$

$$= -H_0 \sum_{i=1}^{T-1} 2^i + H_0 2^T$$

$$= H_0 \left[-\left(\frac{2^T - 1}{2-1} \right) + 2^T \right] = H_0$$

If $P(T < \infty) = 1$ we have ensured $E[W_T] = H_0$

$\neq E[W_0] = E[Y_0 H_0] = 0$

<u>Gambler's ruin</u>

Consider the previous example, but now, let $H_n = 1$. Let $S_0 = k$ represent your initial fortune, and if $S_n = M$, then you bankrupt Mr. House.

Let $T = \inf\{n : S_n = 0 \text{ or } M\}$

Let $A = \{S_T = 0\}$ this is $\mathcal{F}_T$ meas.

$S_n - k$ is a martingale. $|S_{n\wedge T} - k| \leq k + M$, so stopping theorem gives

$$E[S_T - k] = E[S_0 - k] = 0$$

$$\Rightarrow \quad -k\,P(S_T = 0) + M\left(1 - P(S_T = 0)\right) = 0$$

$$\Rightarrow \quad P(S_T = 0) = \frac{M-k}{M}$$

Need to show $P(S_T = 0) + P(S_T = M) = 1$

But this is equivalent to showing $P(T < \infty) = 1$ (which was needed for the stopping theorem anyway)

How to show? $P(0 < S_n < M) = P\left(0 < \frac{S_n}{\sqrt{n}} < \frac{M}{\sqrt{n}}\right)$

$$\leq P\left(0 \leq \frac{S_n}{\sqrt{n}} \leq \epsilon\right) \longrightarrow \int_0^{\epsilon} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx \leq C\epsilon$$

But

$$P(T = +\infty) = P\left(\bigcap_n \{0 < S_n < M\}\right) \leq P(0 < S_n < M)$$

Ballot theorem

Given A receives α votes & B receives β, what's the chance that A always strictly leads B in the vote counting process?

This can be proved in two different ways.

1) Using the reflection principle
2) Using "backward" martingales.

1)

Assume all vote counting sequences are equally likely.
$n = \alpha + \beta$. Let

$$X_i \in \begin{cases} +1 & \text{if vote for } A \\ -1 & \text{if vote for } B \end{cases}$$

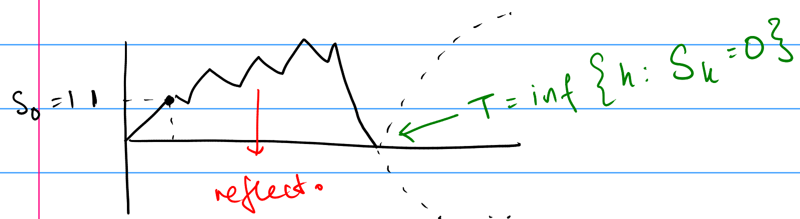
X_i represents the i^{th} vote.
Let $S_k = \sum_{i=1}^{k} X_i$

Find sequences $\vec{X}_n = (X_1 \cdots X_n)$ st $S_k > 0 \ \forall k = 1, \dots, n$.
$(S_k)_{k=1}^{n}$ is clearly a SRW. We need $\alpha > \beta$

$\# \{ \vec{X}_n : S_0 = 0, S_k > 0 \ \forall k, S_n = \alpha - \beta \}$

$= \# \{ \vec{X}_n : S_1 = 1, S_k > 0 \ k = 2, \dots, n, S_n = \alpha - \beta \}$

$= \# \{ \vec{X}_{n-1} \in \{\pm 1\}^{n-1} : S_0 = 1, S_k > 0, k = 1, \dots, n-1, S_{n-1} = \alpha - \beta \}$



Consider the set

$\{\vec{X}_{n-1} : S_0 = 1,\ S_{n-1} = \alpha - \beta\}$. If $T \leq n$, that is, the trajectory hits the y-axis then the trajectory is not good; i.e.,

$\vec{X}_{n-1} \notin \{\vec{X}_{n-1} \in \{\pm 1\}^{n-1} : S_0 = 1,\ S_k > 0,\ k = 1, \ldots, n-1,\ S_{n-1} = \alpha - \beta\}$

But for trajectories in this set, we can replace the initial part of the trajectory with each $(X_1, \ldots X_T)$ replaced by $(-X_1, \ldots, -X_T)$

Then it is easy to see that the reflected trajectory starts at $S_0 = -1$ and ends up at $S_{n-1} = \alpha - \beta$. In fact

$$\#\{\vec{X}_{n-1} : S_0 = 1,\ S_{n-1} = \alpha - \beta\} = \#\{\vec{X}_{n-1} : S_0 = -1,\ S_{n-1} = \alpha - \beta\}$$

In $n-1$ steps of ± 1, we have to end up at $\alpha - \beta + 1$

$$\#\{\text{Up steps}\} - \#\text{down steps} = \alpha - \beta + 1$$

$$a - (n-1-a) = \alpha - \beta + 1 \Rightarrow 2a = n - 1 + \alpha - \beta + 1$$

$$a = \frac{\alpha + \beta + \alpha - \beta}{2} = \alpha.$$

So $$\#\{\vec{X}_{n-1} : S_0 = -1,\ S_{n-1} = \alpha - \beta\} = \binom{n-1}{\alpha}$$

$$\#\{\vec{X}_{n-1} : S_0 = 1,\ S_{n-1} = \alpha - \beta\} = \binom{n-1}{\alpha - 1}$$

one upstep is already taken

$$\#\{\vec{X}_{n-1} : S_0 = 1,\ S_{n-1} = \alpha - \beta,\ T > n\} = \#\{\vec{X}_{n-1} : S_0 = 1,\ S_{n-1} = \alpha - \beta\}$$
$$- \#\{\vec{X}_{n-1} : S_0 = 1,\ S_{n-1} = \alpha - \beta,\ T \leq n\}$$

$$= \binom{n-1}{\alpha-1} - \binom{n-1}{\alpha} = \frac{(n-1)!}{(\alpha-1)!\,\beta!} - \frac{(n-1)!}{\alpha!\,(\beta-1)!}.$$

$$= \frac{(n-1)!}{(\alpha-1)!\,(\beta-1)!}\left[\frac{\alpha-\beta}{\alpha\beta}\right] = \frac{(n-1)!}{\alpha!\,\beta!}(\alpha-\beta)$$

$$\#\{\vec{X}_n, S_0 = 0, S_n = \alpha-\beta\} = \binom{n}{\alpha} = \frac{n!}{\alpha!\,\beta!}.$$

Taking the ratio gives $\frac{\alpha-\beta}{\alpha+\beta}$.

2)

Backward Martingales

Suppose $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ $\quad \mathcal{F}_n = \sigma(S_n, S_{n+1}, \dots)$

Then S_n is $\mathcal{F}_n$ measurable. Let $\tilde{S}_n = \frac{S_n}{n}$

$$E[\tilde{S}_n \mid \mathcal{F}_{n+1}] = E\left[\frac{S_{n+1}}{n} - \frac{X_n}{n} \,\Big|\, \mathcal{F}_{n+1}\right] = \frac{S_{n+1}}{n} - E\left[\frac{X_n}{n} \,\Big|\, \mathcal{F}_{n+1}\right] \text{ — } \star 1$$

$E[X_i \mid \mathcal{F}_{n+1}] = E[X_j \mid \mathcal{F}_{n+1}]$ $\quad i,j \le n+1$ by symmetry.

$$\Rightarrow \sum_{i=1}^{n+1} E[X_i \mid \mathcal{F}_{n+1}] = S_{n+1} \Rightarrow E[X_n \mid \mathcal{F}_{n+1}] = \frac{S_{n+1}}{n+1}$$

$$\star 1 = \frac{S_{n+1}}{n} - \frac{S_{n+1}}{n+1} = \tilde{S}_{n+1}$$

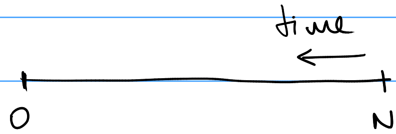
Thus $\tilde{S}_n$ is a backwards martingale satisfying

$$E[\tilde{S}_n \mid \mathcal{F}_{n+1}] = \tilde{S}_{n+1}$$

The stopping Theorem applies to backward martingales.

Let N be some fixed horizon, and let $0 \le T \le N$ be st

$\{T \geq k\}$ is $\mathcal{F}_k$ measurable.



As before $T \vee k$ is a stopping time,

$$E[\tilde{S}_T] = \lim_{k \downarrow 0} E\left[\tilde{S}_{T \vee k}\right] = \tilde{E}[S_N]$$

Remark: There is no centering here to worry about in $\tilde{S}_n$

Let us apply this to Ballots.

Let $X_i \in \{0,2\}$, where 2 represents a vote for B & 0 for A.

$\bar{X}_i = -(X_i - 1) \in \{-1,1\}$ $\quad \{\bar{S}_0 = 0,\ \bar{S}_k > 0,\ k = 1, \dots, N,\ \bar{S}_N = \alpha - \beta\}$

$= \{S_0 = 0,\ -(S_k - k) > 0,\ k = 1, \dots N,\ S_N = (N - (\alpha - \beta))\}$ $\quad \alpha + \beta - \alpha + \beta = 2\beta$

$= \{S_0 = 0,\ k > S_k,\ k = 1, \dots N,\ S_N = 2\beta\}$

$= G$

Let $T = \sup\{0 \le k \le N : S_k \geq k\}$ and $T = 1$ if it doesn't happen.

$\tilde{S}_T = 1$ on G^c. $\quad S_{T+1} < T+1$ $\quad$ Then

$S_T \le S_{T+1} \le T$ since S_{j+1} is integer valued and $X_i \geq 0$.

$\Rightarrow S_T = T \Rightarrow \hat{S}_T = 1$.

On G, $T = 1$, and $S_1 < 1$ (since G happens)

$\Rightarrow S_1 = 0$ (since $X_1 \in \{0,2\}$)

Therefore $S_T = 0$ on G.

Then $\tilde{\hat{S}}_T = 1_G$

$$E[1_G] = E[\tilde{\hat{S}}_T] = E[\tilde{\hat{S}}_N] = \frac{S_N}{N} = \frac{\alpha-\beta}{\alpha+\beta}$$

Crazy, huh? A more general version appears in Durrett.

Thm: (Doob's Maximal inequalities)

If X - subm, then $\forall \lambda > 0$, $\forall n \geq 1$

1) $\circledast$ $\quad \lambda P\left(\max_{1\leq j\leq n} X_j \geq \lambda\right) \leq E[X_n; \max_{1\leq j\leq n} X_j \geq \lambda] \leq E[X_n^+]$

2) $\quad \lambda P\left(\min_{1\leq j\leq n} X_j \leq -\lambda\right) \leq E[X_n^+] - EX_1$

Pf: 1) Let $T = \inf\{j : X_j \geq \lambda\}$. T is a stopping time

$\{T \leq n\} = \max_{1\leq j\leq n} X_j \geq \lambda$.

LHS of $\circledast$ $= \lambda P(T \leq n) = \sum_{j=1}^{n} \lambda P(T=j)$

$\leq \sum_{j=1}^{n} E[X_j; T=j)$

X - subm $\Rightarrow$ $\quad E[X_j; T=j] \leq E[E[X_n | \mathcal{F}_j]; T=j]$

$\{T=j\} \subseteq \mathcal{F}_j$ $\quad \Rightarrow \quad = E[X_n; T=j]$

So LHS of $\circledast$ $\leq \sum_{j=1}^{n} E[X_n; T=j] = E[X_n; T\leq n] \leq EX_n^+$

2) $\tau := \inf\{1 \le j \le n : X_j \le -\lambda\}$. - stopping time.
($\inf \phi = \infty$)

So $\{\min_{1\le j\le n} X_j \le -\lambda\} = \{\tau \le n\}$.

On $\tau \le n$ have $X_\tau \le -\lambda$ so

$E[X_\tau ; \tau \le n] \le -\lambda P(\tau \le n)$

also $E[X_n ; \tau > n] \le E[X_n^+]$

add

$$E X_{\tau\wedge n} \le -\lambda P(\tau \le n) + E[X_n^+]$$

$1 \le \tau \wedge n$ a.s. $\Rightarrow$ by Optional stopping

$$X_1 \le E[X_{\tau\wedge n} ; \mathcal{F}_1]$$

$$E X_1 \le E \text{ ~~(1)~~ } = E X_{\tau \wedge n}$$

$\triangle$

Thm Doob's Martingale cvge thm

Let X be a subm. Suppose

i) X is bdd in $L^1(P)$

or

ii) $X \le 0$ a.s.

Then $\lim_{n\to\infty} X_n$ exists a.s. & is finite a.s.

Pf) Like LLN, 1st do L^2 case, ~~then use truncation~~ ~~for L^1~~

non-neg, L^2-bdd case

1st let's show X_n is Cauchy, so cv in L^2.

$X \ge 0$, L^2-bdd, so

$$\|X_{n+k} - X_n\|_2^2 = \|X_{n+k}\|_2^2 + \|X_n\|_2^2 - 2E[X_{n+k} X_n]$$

$$-2E[X_{n+k}X_n] = -2E[E(X_{n+k} | \mathcal{F}_n) X_n]$$

$$\le \|X_{n+k}\|_2^2 - \|X_n\|_2^2$$

X^2 subm $\Rightarrow$ $\|X_n\|_2 \nearrow \sup_n \|X_n\|_2 < \infty$ since L^2 bdd.

$\Rightarrow \{X_n\}_{n\ge1}^\infty$ Cauchy seq in $L^2 \Rightarrow$ cv in L^2.

Let $X_\infty =$ L^2 limit of X_n

To show $X_n \to X_\infty$ a.s., use Borel-Cantelli

Need $\sum_{k\geq 1}^{\infty} P(|X_\infty - X_n| > \varepsilon) < \infty$. Might not have it. Work w/ a subseq 1st.

$\exists n_k \uparrow \infty$ s.t. $\|X_\infty - X_{n_k}\|_2 \leq 2^{-k}$.

Then $\sum_{k\geq 1}^{\infty} P(|X_\infty - X_{n_k}| \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\sum 4^{-k} < \infty$ so $X_{n_k} \to X_\infty$ a.s.

Let's show X_j w/ $n_k \leq j \leq n_{k+1}$ are close to X_{n_k}.

Consider $\max_{n_k \leq j \leq n_{k+1}} |X_j - X_{n_k}|$.

Use Doob's max'l ineq. Apply to $X_j - X_{n_k} = X_{n_k + \ell} - X_{n_k}$

$\{X_{n_k+\ell} - X_{n_k}\}_{\ell \geq 0}^{\infty}$ submartingale $\Rightarrow$ by Doob's ineq.

(combine both parts)

$$P\left(\max_{n_k \leq j \leq n_{k+1}} |X_j - X_{n_k}| > \varepsilon\right) \leq \frac{2}{\varepsilon} E|X_{n_{k+1}} - X_{n_k}| - \frac{1}{\varepsilon} E 0$$

$$\leq \frac{2}{\varepsilon}\|X_{n_{k+1}} - X_{n_k}\|_2 \leq \frac{2}{\varepsilon}\left(\|X_{n_{k+1}} - X_\infty\|_2 + \|X_{n_k} - X_\infty\|_2\right)$$

$$\leq \frac{2}{\varepsilon}(2^{-k} + 2^{-k-1}) = \frac{3}{\varepsilon} 2^{-k}$$

Since this is summable in k we have by Borel-Cantelli

$$\max_{n_k \leq j \leq n_{k+1}} |X_j - X_{n_k}| \xrightarrow[k\to\infty]{} 0 \quad \text{a.s.}$$

Combined w/ $X_{n_k} \to X_\infty$ a.s. get $X_n \to X_\infty$ a.s.

$X_\infty \in L^2(P) \Rightarrow X_\infty$ is a.s. finite.

<u>non-positive case</u>

If $X_n \leq 0$ subm, $\Rightarrow e^{X_n}$ bdd non-neg subm $\Rightarrow$ by prev. case $e^{X_n} \to e^{X_\infty}$ a.s. w/ e^{X_∞} finite a.s.

<u>non-neg L^1-bdd case</u>

$X_n \geq 0$, subm, bdd in $L^1(P) \Rightarrow$ by Krickeberg decomp.

$X_n = Y_n - Z_n$, Y_n - martingale, Z_n - non-neg superm.

Know pf of Krickeberg $Y_n \geq 0$.

Both Y_n, Z_n - L^1-bdd

$-Y_n$ - non-pos. subm $\Rightarrow$ by prev. step $-Y_n$ cv a.s. to a.s. finite limit.

$-Z_n$ - ———— // ————

<u>L^1-bdd case</u>

$$X_n = Y_n - Z_n \qquad Y \ L^1\text{-bdd } m, \ Z - \text{non-neg } L^1\text{-bdd super-}$$

as in prev. case Z_n cv a.s. to a.s. finite limit.

$Y_n = Y_n^+ - Y_n^-$, Y_n^+, Y_n^- - non-neg L^1-bdd subm $\Rightarrow$ by prev. done. $\triangle$

Suggest reading applications of Martingales

e.g. Pf of Kolmogorov's 0-1 law

Lévy's Borel-Cantelli

<u>Khintchine's law of iterated logarithms</u>

$\{X_i\}_{i=1}^\infty$ iid $\qquad S_n = X_1 + \dots + X_n$

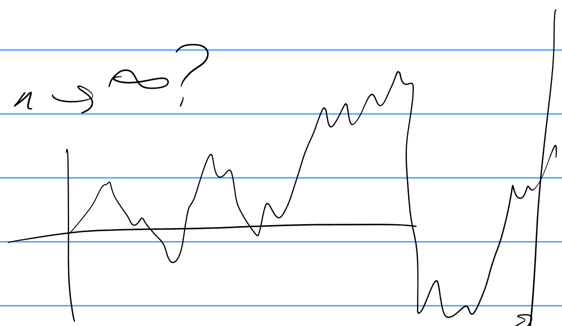
mean μ

Var σ^2

How does S_n behave as $n \to \infty$?

$\frac{S_n}{n} \xrightarrow{a.s.} \mu$

$\frac{S_n - n\mu}{\sqrt{n}} \Rightarrow N(0,1)$.



So typically S_n of order $\sqrt{n}$ away from the mean

Slice far away distr ~ normal

Q: How large does it get?

Khintchine's Law of the iterated logarithm

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \ln \ln n}} = 1 \text{ a.s.} \quad (\text{for liminf get } -1)$$

<u>Pf in book.</u>

Another big application - Stock market
Option pricing
Black-Scholes

Rademacher's thm

I - any interval $\subseteq \mathbb{R}$

<u>Def'n</u>: A fcn $f: I \to \mathbb{R}$ is Lipschitz if $\exists$ const $A > 0$ s.t. $\forall x, y \in \mathbb{R}$

$$|f(x) - f(y)| \le A|x-y|.$$

The optimal A is called the Lipschitz const of f.
(Can define for f).

Lipschitz $\Rightarrow$ Cts.

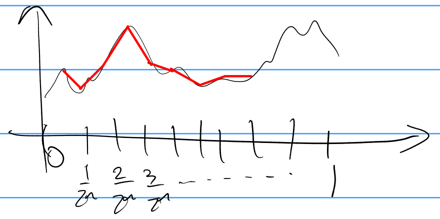
diff'ble w/ cts derivative on cpt set $\Rightarrow$ Lipschitz

<u>converse</u>

In pf: if f has deriv, f'=g, and g int'l, then can write f(x)-f(0)=int_0^x g(t)dt. Let's try to write f as such integral.If g is cts, then FundThmCalc tells us f diff'le. We won't have g diff'le, but only int'le. Lebesgue differentiation thm gives that integrals of L1 functions are almost everywhere differentiable. What's a good candidate for g? Take approximation of g by finite difference quotients.

<u>Thm</u> (Rademacher)
If $f: I \to \mathbb{R}$ is Lipschitz where I is an interval, then f is diff'le almost everywhere.

<u>Pf</u>: Let $I = [0,1]$, P - the Leb mre on $([0,1], \mathcal{B}([0,1]))$
P - prob mre.



Divide $[0,1]$ into 2^n parts & consider the slopes on each:

$$\frac{f((j+1)2^{-n}) - f(j2^{-n})}{(j+1)2^{-n} - j2^{-n}} \quad j \in \{0, 1, \ldots, 2^n - 1\}.$$

Let $\mathcal{F}_n^0 = \{ (j2^{-n}, (j+1)2^{-n}],\ j \in \{0, \ldots, 2^n - 1\}\}$ - dyadic intervals

$\mathcal{F}_n = \sigma(\mathcal{F}_n^0)$

$\mathcal{F} = \{\mathcal{F}_n\}_{n \ge 1}$ - dyadic filtration

Given $Q \in \mathcal{F}_n^0$, let $l(Q)$ be the left endpt
$r(Q)$ — " — right — " —.

Define
$$X_n(\omega) := \sum_{Q \in \mathcal{F}_n^0} \frac{f(r(Q)) - f(l(Q))}{r(Q) - l(Q)} \mathbb{1}_{Q}(\omega) \quad \forall \omega \in (0,1].$$

<u>Rmk</u>: notice that $\forall \omega \in (0,1]$ exactly one summand is nonzero.

Rmk 2: Given $w \in (0,1]$, $\{X_n(w)\}_n$ is a sequence of difference quotients approximating what would be $f'(w)$.

Claim: X is a martingale wrt $\mathcal{F}$.

Pf: NTS $E[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}$.

Write X_n into $\mathcal{F}^0_{n-1}$

$$X_n = \sum_{J \in \mathcal{F}^0_{n-1}} \sum_{\substack{Q \in \mathcal{F}^0_n \\ Q \subseteq J}} \frac{f(r(Q)) - f(\ell(Q))}{2^{-n}} \mathbb{1}_Q(w)$$

For $Q \subseteq J$ as in the sum $E[\mathbb{1}_Q \mid \mathcal{F}_{n-1}] = \frac{1}{2} \mathbb{1}_J$

so

$$E[X_n \mid \mathcal{F}_{n-1}] = \frac{1}{2} \sum_{J \in \mathcal{F}^0_{n-1}} \sum_{\substack{Q \in \mathcal{F}^0_n \\ Q \subseteq J}} \frac{f(r(Q)) - f(\ell(Q))}{2^{-n}} \mathbb{1}_J(w) = X_{n-1}.$$

So X - Martingale wrt $\mathcal{F}$.

X - bdd $\Rightarrow$ by UI thm $\exists$ RV X_∞ s.t.

$X_n \to X_\infty$ P almost surely & in $L^1(P)$ $\triangle$ claim

Claim: $f(x) - f(0) = \int_0^x X_\infty(w)\,dw \quad \forall x \in (0,1]$.

Pf: Given $x \in (0,1]$, $\exists!\, J \in \mathcal{F}^0_n$ st. $x \in J$.

Then $|f(x) - f(r(J))| \le \underset{\substack{\uparrow \\ \text{Lipschitz const of } f}}{A} |x - r(J)| \le \frac{A}{2^n}$ (1)

$$f(r(J)) - f(0) = \sum_{\substack{L \in \mathcal{F}^0_n \\ L \leq J}} f(r(L)) - f(\ell(L)) \qquad (2)$$

$$= \sum_{\substack{L \in \mathcal{F}^0_n \\ L \leq J}} \int_L X_n(w)\,dw = \int_0^{r(J)} X_n(w)\,dw$$

Now $\left| \int_0^{r(J)} X_n(w)\,dw - \int_0^x X_n(w)\,dw \right| \le \int_x^{x+2^{-n}} |X_n(w)|\,dw \xrightarrow[\text{by DCT}]{n\to\infty} 0$ (3)

So combining (1), (2), (3) get $\lim_{n\to\infty} f(x) - f(0) - \int_0^x X_n(w)\,dw = 0.$

Now $\left|\int_0^x X_n(u) - X_\infty(u)\,du\right| \leq \int_0^x |X_n(u) - X_\infty(u)|\,du \leq \int_0^1 |X_n(u) - X_\infty(u)|\,du \to 0$

so $f(x) - f(0) = \int_0^x X_\infty(u)\,du \quad \forall x \in (0,1]$. Δ Claim.

If X_∞ were cts, could use the fundamental thm of calc to say f is diff'ble everywhere and $f' = X_\infty$.

Only know $X_\infty \in L^1(P)$.

Generalisation of FTC, Lebesgue's differentiation thm (see pf in section 8.6.4) says

if $\int_0^1 |X_\infty(u)|\,du < \infty$, then $\lim_{\delta \downarrow 0} \frac{1}{\delta}\int_\omega^{\omega+\delta} X_\infty(y)\,dy = X_\infty(y)$ almost every $\omega \in (0,1]$

i.e. $\frac{f(x+\delta) - f(x)}{\delta} \to X_\infty(x)$ // Δ

Random patterns

Let $X_1, X_2, \ldots$ be iid $P(X_i = 1) = p \in (0,1)$.

$P(X_i = 0) = q = 1-p$.

$X_1, X_2, X_3, \ldots$ random seq of 0s & 1's.

Q: How long do you need to wait for the 1st 0?

$N = \inf\{k : X_1, \ldots, X_k \text{ contains a } 0\}$.

$EN = ?$

More generally, fix any deterministic pattern.

e.g. $P = 011001$

$N_P := \inf\{k : X_1, \ldots, X_k \text{ contains the pattern } P\}$

$EN_P = ?$

Easy to check $N_P < \infty$ a.s. $\forall P$.

i.e. $\forall$ pattern appears a.s.

E.g. $P(N_0 = j) = p^{j-1}q$ so $EN_0 = \sum_{j=1}^\infty j p^{j-1} q = \frac{1}{q}$

Hard to replicate even for N_{01}.

Let $Y_n := \frac{1}{q} 1_{\{X_1=0\}} + \cdots + \frac{1}{q} 1_{\{X_n=0\}} = \frac{\#\,0\text{s up to } n}{q}$

$\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$.

$E[Y_{n+1} \mid \mathcal{F}_n] = Y_n + \frac{1}{q} P(X_{n+1}=0 \mid \mathcal{F}_n) = Y_n + 1$

So $\{Y_n - n\}_{n\geq 1}^{\infty}$ - mean zero martingale

$\Rightarrow$ by the optional stopping thm

$E(Y_{N \wedge n} - N \wedge n) = 0$.

So $E Y_{N\wedge n} = E N_{\wedge n}$.

$N < \infty$ a.s. & both $\nearrow$ increasing

so can apply MCT, $n \to \infty$ get $E Y_N = E N$.

$Y_N = \frac{1}{q}$ a.s. so $EN = \frac{1}{q}$

This generalizes.

E.g. $E N_{\underbrace{010}_{T}}$ - ? $\quad Z_n := \frac{1}{qpq} 1_{(X_1,X_2,X_3)=T} + \frac{1}{qpq} 1_{(X_2,X_3,X_4)=T} + \cdots + \frac{1}{qpq} 1_{(X_{n-2},X_{n-1},X_n)=T}$

$+ \frac{1}{qp} 1_{(X_{n-1},X_n)=(01)} + \frac{1}{q} 1_{X_n=0}$.

$$E[Z_{n+1} - Z_n \mid \mathcal{F}_n] = E\Big[\frac{1}{qpq} 1_{(X_{n-1},X_n,X_{n+1})=(010)} + \frac{1}{qp} 1_{(X_n,X_{n+1})=(01)} + \frac{1}{q} 1_{X_{n+1}=0}$$

$$- \frac{1}{qp} 1_{(X_{n-1},X_n)=(01)} - \frac{1}{q} 1_{X_n=0} \,\Big|\, \mathcal{F}_n\Big]$$

$$= \cancel{\frac{1}{qpq} 1_{(X_{n-1},X_n)=(01)} \underbrace{E[1_{X_{n+1}=0} \mid \mathcal{F}_n]}_{P(X_{n+1}=0)=q}}$$

$$+ \cancel{\frac{1}{qp} 1_{X_n=0} \underbrace{E[1_{X_{n+1}=1} \mid \mathcal{F}_n]}_{P(X_{n+1}=1)=p}}$$

$$+ \frac{1}{q} \underbrace{E[1_{X_{n+1}=0} \mid \mathcal{F}_n]}_{P(X_{n+1}=0)=q} - \cancel{\frac{1}{qp} 1_{(X_{n-1},X_n)=(01)}} - \cancel{\frac{1}{q} 1_{X_n=0}}$$

$= 1$.

So $(Z_n - n)$ - martingale w/ mean 0.

$\Rightarrow \quad EN_{010} = EZ_N = \frac{1}{qpq} + \frac{1}{q}.$

Similarly $EN_{001} = \frac{1}{qqp}$: $W_n = \sum_{i=3}^{n} \frac{1}{qqp} 1_{(X_{i-2},X_{i-1},X_i)=001} + \frac{1}{qq} 1_{(X_{n-1},X_n)=(00)} + \frac{1}{q} 1_{X_n=0}.$

Q) Which pattern comes 1st, 010 or 011?

$P(010 \text{ before } 001) = ?$

$Z_n - W_n$ — Mean 0 martingale

Let $T = \inf\{k: X_1,\dots,X_k \text{ contains } 010 \text{ or } 011\}$.

Optional stopping & MCT $E[Z_T - W_T] = 0$.

$$Z_T - W_T = \begin{cases} \frac{1}{qpq} + \frac{1}{qp} + \frac{1}{q} - \frac{1}{q} & \text{if } 010 \text{ comes 1st} \\ -\left(\frac{1}{qqp} + \frac{1}{qq} + \frac{1}{q} - \frac{1}{q}\right) & \text{if } 001 \text{ comes 1st} \end{cases}$$

So $E[T_T - W_T] = \underbrace{P(010 \text{ 1st})}_{s} \cdot \left(\frac{1}{qpq} + \frac{1}{qp}\right) - \underbrace{P(011 \text{ 1st})}_{1-s}\left(\frac{1}{qqp} + \frac{1}{qq}\right) = 0$

$$P(010 \text{ before } 011) = s = \frac{\frac{1}{qpp} + \frac{1}{qq}}{\frac{1}{qpp} + \frac{1}{qq} + \frac{1}{qpq} + \frac{1}{qp}} = \frac{\frac{1}{pp} + \frac{1}{q}}{\frac{1}{pp} + \frac{1}{q} + \frac{1}{pq} + \frac{1}{p}} = \frac{q + p^2}{q + p^2 + p + pq}$$

$$= \frac{p^2 - p + 1}{}$$